Central limit theorem for an additive functional of the fractional Brownian motion II

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Abstract

We prove a central limit theorem for an additive functional of the d-dimensional fractional Brownian motion with Hurst index $H \in (\frac{1}{2+d}, \frac{1}{d})$, using the method of moments, extending the result by Papanicolaou, Stroock and Varadhan in the case of the standard Brownian motion.

Keywords: fractional Brownian motion, central limit theorem, local time, method of moments.

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1 Introduction

Let $\{B(t) = (B^1(t), \dots, B^d(t)), t \geq 0\}$ be a d-dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. The local time of B, defined as $L_t(x) = \int_0^t \delta(B(s) - x) ds$, for $t \geq 0$ and $x \in \mathbb{R}^d$, where δ is the Dirac delta function, exists and is jointly continuous in t and x if $Hd < 1$ (see [2]). For any integrable function $f : \mathbb{R}^d \to \mathbb{R}$, using the scaling property of the fBm and the continuity of the local time, one can easily show the following convergence in law in the space $C([0,\infty))$, as n tends to infinity

$$\left(n^{Hd-1} \int_0^{nt} f(B(s))\, ds\,, t \geq 0 \right) \overset{\mathcal{L}}{\to} \left(L_t(0) \int_{\mathbb{R}^d} f(x)\, dx\,, t \geq 0 \right). \tag{1.1}$$

If we assume that $\int_{\mathbb{R}^d} f(x)\, dx = 0$, a central limit theorem holds with a random variance. In order to formulate this theorem, we need to introduce some notation. Fix a number $\beta > 0$ and denote

$$H_0^\beta = \left\{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)||x|^\beta dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} f(x)\, dx = 0 \right\}.$$

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For any $f \in H_0^\beta$, and assuming $\beta \in (0,2)$, the quantity (see Lemma 4.1 in [3])

$$\|f\|_\beta^2 := -\int_{\mathbb{R}^{2d}} f(x)f(y)|x-y|^\beta \, dx\, dy = c_{\beta,d} \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-\beta-d} dx \tag{1.2}$$

is finite and nonnegative, where $\widehat{f}$ denotes the Fourier transform of f.

Then, the following central limit theorem holds.

Theorem 1.1 *Suppose $\frac{1}{d+2} < H < \frac{1}{d}$ and $f \in H_0^{\frac{1}{H}-d}$. Then*

$$\Big(n^{\frac{Hd-1}{2}} \int_0^{nt} f(B(s))\, ds\,,\ t \geq 0\Big) \quad \xrightarrow{\mathcal{L}} \quad \Big(\sqrt{C_{H,d}}\, \|f\|_{\frac{1}{H}-d} W(L_t(0))\,, t \geq 0\Big)$$

in the space $C([0,\infty))$, as n tends to infinity, where W is a real-valued standard Brownian motion independent of B and

$$C_{H,d} = \frac{2}{(2\pi)^{\frac{d}{2}}} \int_0^\infty w^{-Hd}\big(1-\exp(-\frac{1}{2w^{2H}})\big)\, dw = \frac{2^{1-\frac{1}{2H}}}{(1-Hd)\pi^{\frac{d}{2}}} \Gamma\Big(\frac{Hd+2H-1}{2H}\Big).$$

This theorem has been proved by Hu, Nualart and Xu in the reference [3], in the case where the Hurst parameter H satisfies $\frac{1}{d+1} < H < \frac{1}{d}$, and it has been conjectured in that paper that the result can be extended to the case $\frac{1}{d+2} < H \leq \frac{1}{d+1}$. The purpose of the present paper is to prove this conjecture. With this aim we will develop a new approach to prove Theorem 1.1 based on Fourier analysis.

Note that the lower bound $\frac{1}{d+2}$ is optimal because for $H \leq \frac{1}{d+2}$ the constant $C_{H,d}$ is infinite. When $d = 1$ and $H = \frac{1}{2}$, the above theorem was obtained by Papanicolaou, Stroock and Varadhan in [4] with $C_{\frac{1}{2},1} = 2$.

As in the reference [3], the proof of Theorem 1.1 is based on the method of moments. In order to handle the integrals on $[0,t]^{2m}$, with respect to the measure $ds_1 \cdots ds_{2m}$, we make the change of variables $u_{2k-1} = n(s_{2k} - s_{2k-1})$ and $u_{2k} = s_{2k}$, $1 \leq k \leq m$. Then, the increments of B in small intervals will be responsible for the independent noise appearing in the limit. The main novelty of our approach, in comparison with [3], is a new methodology based on Fourier analysis and an iterative procedure in order to get the right estimates to derive the tightness of the laws and to show the convergence to zero in the truncation argument.

After some preliminaries in Section 2, in Section 3 we prove some technical estimates based on Fourier analysis which play a fundamental role in our approach. Finally, Section 4 is devoted to the proof of Theorem 1.1. Throughout this paper, if not mentioned otherwise, the letter c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line.

2 Preliminaries

Let $\big\{B(t) = (B^1(t), \dots, B^d(t)), t \geq 0\big\}$ be a d-dimensional fractional Brownian motion with Hurst index $H \in (0,1)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, the compo-

nents of B are independent centered Gaussian processes with covariance

$$\mathbb{E}\left(B^i(t)B^i(s)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The next lemma (see Lemma 2.1 in [3]) gives a formula for the moments of the increments of the process $\{W(L_t(0)), t \geq 0\}$ on disjoint intervals, where W is a real-valued standard Brownian motion independent of B.

Lemma 2.1 *Fix a finite number of disjoint intervals $(a_i, b_i]$ in $[0,\infty)$, where $i = 1,\dots,N$ and $b_i \leq a_{i+1}$. Consider a multi-index $\mathbf{m} = (m_1,\dots,m_N)$, where $m_i \geq 1$ and $1 \leq i \leq N$. Then*

$$\mathbb{E}\Big(\prod_{i=1}^N \big[W(L_{b_i}(0)) - W(L_{a_i}(0))\big]^{m_i}\Big) \tag{2.1}$$
$$= \begin{cases} \Big(\prod_{i=1}^N \frac{m_i!}{2^{\frac{m_i}{2}}(2\pi)^{\frac{m_i d}{4}}(m_i/2)!}\Big)\int_{\prod_{i=1}^N [a_i,b_i]^{\frac{m_i}{2}}} \det(A(w))^{-\frac{1}{2}}\, dw & \textit{if all } m_i \textit{ are even} \\ 0 & \textit{otherwise,} \end{cases}$$

where $A(w)$ is the covariance matrix of the Gaussian random vector

$$\Big(B(w_k^i) : 1 \leq i \leq N \text{ and } 1 \leq k \leq \frac{m_i}{2}\Big).$$

As a consequence, the law of the random vector $\big(W(L_{b_i}(0)) - W(L_{a_i}(0)) : 1 \leq i \leq N\big)$ is determined by the moments computed in the above lemma.

We shall use the following local nondeterminism property of the fractional Brownian motion (see [1]): For any $n \geq 2$ there exists a positive constant k_H depending on n, such that for any $0 = s_0 < s_1 \leq \cdots \leq s_n < \infty$ and $u_1,\dots,u_n \in \mathbb{R}^d$,

$$\operatorname{Var}\Big(\sum_{i=1}^n u_i \cdot \big(B(s_i) - B(s_{i-1})\big)\Big) \geq k_H \sum_{i=1}^n |u_i|^2 (s_i - s_{i-1})^{2H}. \tag{2.2}$$

3 Technical estimates

We are interested in the sequence of stochastic processes defined by

$$F_n(t) = n^{\frac{Hd-1}{2}} \int_0^{nt} f(B(s))\, ds.$$

For $0 \leq a < b < \infty$ and $m \in \mathbb{N}$, let $I_m^n = \mathbb{E}\left[(F_n(b) - F_n(a)^m\right]$. It is easy to see that

$$\begin{aligned} I_m^n &= m!\, n^{m\frac{Hd-1}{2}} \int_{D_m} \mathbb{E}\Big(\prod_{i=1}^m f(B(s_i))\Big)\, ds \\ &= c_{m,d}\, n^{m\frac{Hd-1}{2}} \int_{\mathbb{R}^{md}} \int_{D_m} \Big(\prod_{i=1}^m \widehat{f}(y_i)\Big) \exp\Big(-\frac{1}{2}\operatorname{Var}\big(\sum_{i=1}^m y_i \cdot B(s_i)\big)\Big)\, ds\, dy, \end{aligned}$$

where $c_{m,d} = \frac{m!}{(2\pi)^{md}}$ and $D_m = \{(s_1,\dots,s_m) : na < s_1 < \dots < s_m < nb\}$. Making the change of variables $x_i = \sum_{j=i}^m y_j$ (with the convention that $x_{m+1}=0$) we can write

$$I_m^n = c_{m,d}\, n^{m\frac{Hd-1}{2}} \int_{\mathbb{R}^{md}} \int_{D_m} \Big(\prod_{i=1}^m \widehat{f}(x_i - x_{i+1}) \Big) \\ \times \exp\Big(-\frac{1}{2} \mathrm{Var}\big(\sum_{i=1}^m x_i \cdot (B(s_i) - B(s_{i-1})) \big) \Big)\, ds\, dx.$$

The main idea in order to estimate these terms is to replace each product $\widehat{f}(x_{2i-1}-x_{2i})\widehat{f}(x_{2i}-x_{2i+1})$ by $\widehat{f}(-x_{2i})\widehat{f}(x_{2i}) = |\widehat{f}(x_{2i})|^2$. Then, the differences $\widehat{f}(x_{2i-1}-x_{2i}) - \widehat{f}(-x_{2i})$ and $\widehat{f}(x_{2i}-x_{2i+1}) - \widehat{f}(x_{2i})$ are bounded by constant multiples of $|x_{2i-1}|^\alpha$ and $|x_{2i+1}|^\alpha$, respectively, for any $0 \le \alpha \le 1$, because $\widehat{f}(0)=0$ due to the fact that f has zero integral. We are going to make these substitutions recursively. To do this, we introduce the following notation.

Let $I_{m,0}^n = I_m^n$. For $k = 1,\dots,m$, we define

$$I_{m,k}^n = c_{m,d}\, n^{m\frac{Hd-1}{2}} \int_{\mathbb{R}^{md}} \int_{D_m} I_k \prod_{i=k+1}^m \widehat{f}(x_i - x_{i+1}) \\ \times \exp\Big(-\frac{1}{2} \mathrm{Var}\big(\sum_{i=1}^m x_i \cdot (B(s_i) - B(s_{i-1})) \big) \Big)\, ds\, dx,$$

where

$$I_k = \begin{cases} \prod_{j=1}^{\frac{k-1}{2}} |\widehat{f}(x_{2j})|^2 \widehat{f}(-x_{k+1}), & \text{if } k \text{ is odd;} \\ \prod_{j=1}^{\frac{k}{2}} |\widehat{f}(x_{2j})|^2, & \text{if } k \text{ is even.} \end{cases}$$

The following proposition controls the difference between $I_{m,k-1}^n$ and $I_{m,k}^n$. We fix a positive constant γ such that

$$\gamma < \begin{cases} \frac{1-Hd}{2} & \text{if } 1-Hd \le H; \\ \frac{2H-1-Hd}{2} & \text{if } H < 1-Hd < 2H. \end{cases} \tag{3.1}$$

Proposition 3.1 *For $k = 1,2,\dots,m$, there exists a positive constant c, which depends on γ, such that*

$$|I_{m,k-1}^n - I_{m,k}^n| \le c\, n^{-\gamma} (b-a)^{m\frac{1-Hd}{2}-\gamma}.$$

Proof. The proof will be done in several steps.

Step 1. Suppose first that $k=1$. Applying the local nondeterminism property (2.2) and making the change of variable $u_1 = s_1$, $u_i = s_i - s_{i-1}$, for $2 \le i \le m$, we can show that $|I_{m,0}^n - I_{m,1}^n|$ is less than a constant multiple of

$$n^{m\frac{Hd-1}{2}} \int_{\mathbb{R}^{md}} \int_{O_m} \big|\widehat{f}(x_1 - x_2) - \widehat{f}(-x_2)\big| \Big(\prod_{i=2}^m |\widehat{f}(x_i - x_{i+1})| \Big) \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 u_i^{2H} \Big)\, du\, dx,$$

where

$$O_m = \big\{(u_1, \dots, u_m) : 0 < u_i < n(b-a),\, i = 1, \cdots, m\big\}.$$

Taking into account that that $|\widehat{f}(x)| \le c_\alpha |x|^\alpha$ for $\alpha \in [0,1]$, we obtain

$$\begin{aligned} |I^n_{m,0} - I^n_{m,1}| &\le c_1\, n^{m\frac{Hd-1}{2}} \int_{\mathbb{R}^{md}} \int_{O_m} |x_1|^{\alpha_1} \prod_{i=2}^{m-1} \left(|x_i|^{\alpha_i} + |x_{i+1}|^{\alpha_i}\right) |x_m|^{\alpha_m} \\ &\qquad\qquad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 u_i^{2H}\Big)\, du\, dx \\ &= c_1\, n^{m\frac{Hd-1}{2}} \sum_S \int_{\mathbb{R}^{md}} \int_{O_m} |x_1|^{\alpha_1} \prod_{i=2}^{m-1} \left(|x_i|^{p_i\alpha_i} |x_{i+1}|^{\overline{p}_i\alpha_i}\right) |x_m|^{\alpha_m} \\ &\qquad\qquad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 u_i^{2H}\Big)\, du\, dx, \end{aligned} \tag{3.2}$$

where $S = \big\{p_i, \overline{p}_i : p_i \in \{0,1\}, p_i + \overline{p}_i = 1,\, i = 2, \dots, m-1\big\}$ and the α_is are constants in $[0,1]$.

Rewriting the right hand side of (3.2) gives

$$\begin{aligned} |I^n_{m,0} - I^n_{m,1}| \le c_1\, n^{m\frac{Hd-1}{2}} \sum_S \int_{\mathbb{R}^{md}} \int_{O_m} |x_1|^{\alpha_1} |x_2|^{p_2\alpha_2} \Big(\prod_{i=3}^{m-1} |x_i|^{\overline{p}_{i-1}\alpha_{i-1} + p_i\alpha_i}\Big) |x_m|^{\overline{p}_{m-1}\alpha_{m-1} + \alpha_m} \\ \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 u_i^{2H}\Big)\, du\, dx. \end{aligned}$$

Integrating with respect to x gives

$$\begin{aligned} |I^n_{m,0} - I^n_{m,1}| \le c_2\, n^{m\frac{Hd-1}{2}} \sum_S \int_{O_m} u_1^{-Hd - H\alpha_1} u_2^{-Hd-Hp_2\alpha_2} \prod_{i=3}^{m-1} u_i^{-Hd - H(\overline{p}_{i-1}\alpha_{i-1} + p_i\alpha_i)} \\ \times u_m^{-Hd - H(\overline{p}_{m-1}\alpha_{m-1} + \alpha_m)}\, du. \end{aligned}$$

Assume that

$$1 - Hd - H\alpha_1 > 0,\; 1 - Hd - Hp_2\alpha_2 > 0,\; 1 - Hd - H(\overline{p}_{m-1}\alpha_{m-1} + \alpha_m) > 0$$

and

$$1 - Hd - H(\overline{p}_{i-1}\alpha_{i-1} + p_i\alpha_i) > 0 \quad \text{for } i = 3, \dots, m-1.$$

Then

$$|I^n_{m,0} - I^n_{m,1}| \le c_3\, n^{m\frac{Hd-1}{2}} (nb - na)^{m(1-Hd) - H\sum\limits_{i=1}^m \alpha_i}.$$

Fix $\epsilon > 0$. We choose $\alpha_1 = \frac{1-Hd}{H} - \epsilon$ if $1 - Hd \le H$. Otherwise, we let $\alpha_1 = 1$. For $i = 2, \dots, m$, we choose $\alpha_i = \frac{1-Hd}{2H} - \epsilon$. With these choices of α_is, we obtain

$$m\frac{Hd-1}{2} + m(1-Hd) - H\sum_{i=1}^m \alpha_i = \begin{cases} \frac{Hd-1}{2} + mH\epsilon & \text{if } 1 - Hd \le H; \\ \frac{1-Hd-2H}{2} + (m-1)H\epsilon & \text{if } 1 - Hd > H. \end{cases}$$

Thus we can choose ϵ such that $m\frac{Hd-1}{2}+m(1-Hd)-H\sum_{i=1}^m \alpha_i=-\gamma$, and

$$|I^n_{m,0}-I^n_{m,1}|\le c_4\, n^{-\gamma}(b-a)^{m\frac{1-Hd}{2}-\gamma},$$

which is the desired estimation.

Step 2: Suppose now that $k=2$. By the definition of $I^n_{m,1}$ and $I^n_{m,2}$, $|I^n_{m,1}-I^n_{m,2}|$ is less than a constant multiple of

$$n^{m\frac{Hd-1}{2}}\int_{\mathbb{R}^{md}}\int_{O_m}|\widehat{f}(-x_2)||\widehat{f}(x_2-x_3)-\widehat{f}(x_2)|\prod_{i=3}^m|\widehat{f}(x_i-x_{i+1})|\,\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|x_i|^2u_i^{2H}\Big)\,du\,dx.$$

Using similar arguments as in Step 1,

$$\begin{aligned}|I^n_{m,1}-I^n_{m,2}|&\le c_5\, n^{m\frac{Hd-1}{2}}\sum_{S_1}\int_{\mathbb{R}^{md}}\int_{O_m}|x_2|^{\alpha_1}|x_3|^{\alpha_2}\Big(\prod_{i=3}^{m-1}|x_i|^{p_i\alpha_i}|x_{i+1}|^{\overline{p}_i\alpha_i}\Big)|x_m|^{\alpha_m}\\&\qquad\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|x_i|^2u_i^{2H}\Big)\,du\,dx\\&\le c_6\sum_{S_1}n^{m\frac{Hd-1}{2}}\int_{O_m}|u_1|^{-Hd}|u_2|^{-Hd-H\alpha_1}|u_3|^{-Hd-H(\alpha_2+p_3\alpha_3)}\\&\qquad\times\Big(\prod_{i=4}^{m-1}|u_i|^{-Hd-H(\overline{p}_{i-1}\alpha_{i-1}+p_i\alpha_i)}\Big)|u_m|^{-Hd-H(\overline{p}_{m-1}\alpha_{m-1}+\alpha_m)}\,du,\end{aligned}$$

where $S_1=\big\{p_i,\,\overline{p}_i: p_i\in\{0,1\}, p_i+\overline{p}_i=1,\, i=3,\cdots,m-1\big\}$. Then we can conclude as in Step 1.

Step 3: Suppose that k is odd and $3\le k\le m$. Since k is odd, $|I^n_{m,k-1}-I^n_{m,k}|$ is less than a constant multiple of

$$\begin{aligned}&n^{m\frac{Hd-1}{2}}\int_{\mathbb{R}^{md}}\int_{O_m}\Big(\prod_{i=k+1}^m|\widehat{f}(x_i-x_{i+1})|\Big)\\&\qquad\times\big|\widehat{f}(x_k-x_{k+1})-\widehat{f}(-x_{k+1})\big|\Big(\prod_{j=1}^{\frac{k-1}{2}}|\widehat{f}(x_{2j})|^2\Big)\,\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|x_i|^2u_i^{2H}\Big)\,du\,dx.\end{aligned}$$

Therefore, $|I^n_{m,k-1}-I^n_{m,k}|$ is less than a constant multiple of

$$n^{m\frac{Hd-1}{2}}\int_{\mathbb{R}^{md}}\int_{O_m}\Big(\prod_{i=k+1}^m|\widehat{f}(x_i-x_{i+1})|\Big)|x_k|^{\alpha_k}\Big(\prod_{j=1}^{\frac{k-1}{2}}|\widehat{f}(x_{2j})|^2\Big)\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|x_i|^2u_i^{2H}\Big)\,du\,dx.$$

Integrating with respect to x_is and u_is with $i\le k-1$ gives

$$\begin{aligned}&|I^n_{m,k-1}-I^n_{m,k}|\\&\le c_7\,(b-a)^{\frac{k-1}{2}(1-Hd)}\,n^{(m-k+1)\frac{Hd-1}{2}}\int_{\mathbb{R}^{(m-k+1)d}}\int_{O_{m,k}}\Big(\prod_{i=k+1}^m|\widehat{f}(x_i-x_{i+1})|\Big)|x_k|^{\alpha_k}\\&\qquad\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=k}^m|x_i|^2u_i^{2H}\Big)\,du\,dx,\end{aligned}$$

where $du = du_k \cdots du_m$, $dx = dx_k \cdots dx_m$ and

$$O_{m,k} = \big\{(u_k, \dots, u_m) : 0 \le u_i \le n(b-a), i = k, \dots, m\big\}.$$

Applying Step 1 and then doing some algebra, we can obtain

$$|I^n_{m,k-1} - I^n_{m,k}| \le c_8\, n^{-\gamma}(b-a)^{m\frac{1-Hd}{2}-\gamma}.$$

Step 4: The case when k is even and $4 \le k \le m$ is handled in a similar way. ∎

4 Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in two steps. We first show tightness, and then establish the convergence of moments. Tightness will be deduced from the following inequality.

Proposition 4.1 *For any $0 \le a < b \le t$ and any integer $m \ge 1$,*

$$\mathbb{E}\left[(F_n(b) - F_n(a))^{2m}\right] \le C\,(b-a)^{m(1-Hd)-\gamma},$$

where C is a constant depending only on H, m, d and f.

Proof. Note that $\mathbb{E}\left[(F_n(b) - F_n(a))^{2m}\right] = I^n_{2m,0}$. Applying Proposition 3.1 repeatedly gives

$$I^n_{2m,0} \le c_1\, n^{-\gamma}(b-a)^{m(1-Hd)-\gamma} + c_1\, I^n_{2m,2m}. \tag{4.1}$$

So it suffices to estimate $I^n_{2m,2m}$. By the definition of $I^n_{2m,2m}$, using the same notation as in the proof of Proposition 3.1, we obtain

$$\begin{aligned}
I^n_{2m,2m} &= c_2\, n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_{D_{2m}} \Big(\prod_{j=1}^m |\widehat{f}(x_{2j})|^2\Big) \\
&\qquad \times \exp\Big(-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{2m} x_i \cdot \big(B(s_i) - B(s_{i-1})\big)\Big)\Big)\, ds\, dx \\
&\le c_3\, n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_{O_{2m}} \Big(\prod_{j=1}^m |\widehat{f}(x_{2j})|^2\Big) \exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^{2m} |x_i|^2 u_i^{2H}\Big)\, du\, dx \\
&\le c_4\,(b-a)^{m(1-Hd)}\Big(\int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-\frac{1}{H}}\, dx\Big)^m.
\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2) gives the desired result. ∎

Next we shall prove the convergence of all finite dimensional distributions. That is, we shall prove that the moments of $F_n(t)$ converge to the corresponding ones of $W(L_t(0))$.

Fix a finite number of disjoint intervals $(a_i, b_i]$ with $i = 1, \dots, N$ and $b_i \le a_{i+1}$. Let $\mathbf{m} = (m_1, \dots, m_N)$ be a fixed multi-index with $m_i \in \mathbb{N}$ for $i = 1, \dots, N$. Set $\sum_{i=1}^N m_i = |\mathbf{m}|$

and $\prod_{i=1}^{N} m_i! = \mathbf{m}!$. We need to consider the following sequence of random variables

$$G_n = \prod_{i=1}^{N} (F_n(b_i) - F_n(a_i))^{m_i}$$

and compute $\lim_{n\to\infty} \mathbb{E}(G_n)$. Note that the expectation of G_n can be formulated as

$$\mathbb{E}(G_n) = \mathbf{m}!\, n^{\frac{|\mathbf{m}|(Hd-1)}{2}} \mathbb{E}\Big(\int_{D_{\mathbf{m}}} \prod_{i=1}^{N} \prod_{j=1}^{m_i} f(B(s_j^i))\, ds \Big),$$

where

$$D_{\mathbf{m}} = \big\{ s \in \mathbb{R}^{|\mathbf{m}|} : na_i < s_1^i < \cdots < s_{m_i}^i < nb_i, 1 \le i \le N \big\}. \tag{4.3}$$

Here and in the sequel we denote the coordinates of a point $s \in \mathbb{R}^{|\mathbf{m}|}$ as $s = (s_j^i)$, where $1 \le i \le N$ and $1 \le j \le m_i$.

For simplicity of notation, we define

$$J_0 = \big\{ (i,j) : 1 \le i \le N, 1 \le j \le m_i \big\}.$$

For any (i_1, j_1) and $(i_2, j_2) \in J_0$, we define the following dictionary ordering

$$(i_1, j_1) \le (i_2, j_2)$$

if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 \le j_2$. For any (i,j) in J_0, under the above ordering, (i,j) is the $(\sum_{k=1}^{i-1} m_k + j)$-th element in J_0 and we define $\#(i,j) = \sum_{k=1}^{i-1} m_k + j$.

Proposition 4.2 *Suppose that at least one of the exponents m_i is odd. Then*

$$\lim_{n\to\infty} \mathbb{E}(G_n) = 0.$$

Proof. Using Fourier transform, we see that $\mathbb{E}(G_n)$ is equal to

$$\frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{\frac{|\mathbf{m}|(Hd-1)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \Big(\prod_{i=1}^{N} \prod_{j=1}^{m_i} \widehat{f}(y_j^i) \Big) \exp\Big(-\frac{1}{2} \mathrm{Var}\Big(\sum_{i=1}^{N} \sum_{j=1}^{m_i} y_j^i \cdot B(s_j^i) \Big) \Big)\, ds\, dy.$$

Making the change of variables $x_j^i = \sum_{(\ell,k) \ge (i,j)} y_k^\ell$ for $1 \le i \le N$ and $1 \le j \le m_i$,

$$\mathbb{E}(G_n) = \frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{\frac{|\mathbf{m}|(Hd-1)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \prod_{i=1}^{N} \prod_{j=1}^{m_i} \widehat{f}(x_j^i - x_{j+1}^i)$$
$$\times \exp\Big(-\frac{1}{2} \mathrm{Var}\Big(\sum_{i=1}^{N} \sum_{j=1}^{m_i} x_j^i \cdot \big(B(s_j^i) - B(s_{j-1}^i) \big) \Big) \Big)\, ds\, dx.$$

Applying Proposition 3.1, we obtain

$$\lim_{n\to\infty}\mathbb{E}\,(G_n)=\frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}}\lim_{n\to\infty}n^{\frac{|\mathbf{m}|(Hd-1)}{2}}\int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}}\Big(\prod_{(i,j)\in J_e}|\widehat{f}(x_j^i)|^2\Big)\,I_{|\mathbf{m}|}$$
$$\times\exp\bigg(-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{N}\sum_{j=1}^{m_i}x_j^i\cdot\big(B(s_j^i)-B(s_{j-1}^i)\big)\Big)\bigg)\,ds\,dx,$$

where $J_e=\big\{(i,j)\in J_0:\#(i,j)\text{ is even}\big\}$ and

$$I_{|\mathbf{m}|}=\begin{cases}\widehat{f}(x_{m_N}^N), & \text{if } |\mathbf{m}| \text{ is odd;}\\ 1, & \text{if } |\mathbf{m}| \text{ is even.}\end{cases}$$

It is easy to see that $\lim\limits_{n\to\infty}\mathbb{E}\,(G_n)=0$ when $|\mathbf{m}|$ is odd. We shall show $\lim\limits_{n\to\infty}\mathbb{E}\,(G_n)=0$ when $|\mathbf{m}|$ is even. In this case,

$$\lim_{n\to\infty}\mathbb{E}\,(G_n)=\frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}}\lim_{n\to\infty}n^{\frac{|\mathbf{m}|(Hd-1)}{2}}\int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}}\Big(\prod_{(i,j)\in J_e}|\widehat{f}(x_j^i)|^2\Big)$$
$$\times\exp\bigg(-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{N}\sum_{j=1}^{m_i}x_j^i\cdot\big(B(s_j^i)-B(s_{j-1}^i)\big)\Big)\bigg)\,ds\,dx.$$

Note that the right hand side of the above equality is positive. Using the local nondeterminism property (2.2),

$$\Big|\lim_{n\to\infty}\mathbb{E}\,(G_n)\Big|\le c_1\limsup_{n\to\infty}n^{\frac{|\mathbf{m}|(Hd-1)}{2}}\int_{\mathbb{R}^{|\mathbf{m}|d}}\int_{D_{\mathbf{m}}}\Big(\prod_{(i,j)\in J_e}|\widehat{f}(x_j^i)|^2\Big)$$
$$\times\exp\bigg(-\frac{\kappa_H}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i}|x_j^i|^2(s_j^i-s_{j-1}^i)^{2H}\bigg)\,ds\,dx$$
$$:=c_1\limsup_{n\to\infty}I_n.$$

Assume that m_ℓ is the first odd exponent. Integrating with respect to proper x_j^is and s_j^is gives

$$I_n\le c_2\,n^{Hd-1}\sup_{s_{m_\ell-1}^\ell\in(na_\ell,nb_\ell]}\int_{\mathbb{R}^d}\int_{s_{m_\ell-1}^\ell}^{nb_\ell}\int_{na_{\ell+1}}^{nb_{\ell+1}}|\widehat{f}(x_1^{\ell+1})|^2(s_{m_\ell}^\ell-s_{m_\ell-1}^\ell)^{-Hd}$$
$$\times\exp\Big(-\frac{\kappa_H}{2}|x_1^{\ell+1}|^2(s_1^{\ell+1}-s_{m_\ell}^\ell)^{2H}\Big)\,ds_1^{\ell+1}\,ds_{m_\ell}^\ell\,dx_1^{\ell+1}.$$

Note that $|\widehat{f}(x)| \le c_\alpha |x|^\alpha$ for $\alpha \in [0,1]$. Choosing $\alpha \in (\frac{1-Hd}{2H}, \frac{2-Hd}{2H})$ gives

$$\begin{aligned} I_n &\le c_3\, n^{Hd-1} \sup_{s^\ell_{m_\ell-1} \in (na_\ell, nb_\ell]} \int_{\mathbb{R}^d} \int_{s^\ell_{m_\ell-1}}^{nb_\ell} \int_{na_{\ell+1}}^{nb_{\ell+1}} |x_1^{\ell+1}|^{2\alpha} (s^\ell_{m_\ell} - s^\ell_{m_\ell-1})^{-Hd} \\ &\qquad \times \exp\Big(-\frac{\kappa_H}{2} |x_1^{\ell+1}|^2 (s_1^{\ell+1} - s^\ell_{m_\ell})^{2H} \Big)\, ds_1^{\ell+1}\, ds^\ell_{m_\ell}\, dx_1^{\ell+1} \\ &\le c_4\, n^{Hd-1} \sup_{s^\ell_{m_\ell-1} \in (na_\ell, nb_\ell]} \int_{s^\ell_{m_\ell-1}}^{nb_\ell} \int_{na_{\ell+1}}^{nb_{\ell+1}} (s_1^{\ell+1} - s^\ell_{m_\ell})^{-Hd-2H\alpha} (s^\ell_{m_\ell} - s^\ell_{m_\ell-1})^{-Hd}\, ds_1^{\ell+1}\, ds^\ell_{m_\ell} \\ &\le c_5\, n^{Hd-1} \sup_{s^\ell_{m_\ell-1} \in (na_\ell, nb_\ell]} \int_{s^\ell_{m_\ell-1}}^{nb_\ell} (na_{\ell+1} - s^\ell_{m_\ell})^{1-Hd-2H\alpha} (s^\ell_{m_\ell} - s^\ell_{m_\ell-1})^{-Hd}\, ds^\ell_{m_\ell} \\ &\le c_6\, n^{1-Hd-2H\alpha}, \end{aligned}$$

where we used $b_\ell \le a_{\ell+1}$ in the last inequality. Therefore,

$$\Big| \lim_{n\to\infty} \mathbb{E}\,(G_n) \Big| \le c_6 \lim_{n\to\infty} n^{1-Hd-2H\alpha} = 0.$$

∎

Consider now the convergence of moments when all exponents m_i are even.

Proposition 4.3 *Suppose that all exponents m_i are even. Then*

$$\lim_{n\to\infty} \mathbb{E}\,(G_n) = C_{H,d}^{\frac{|\mathbf{m}|}{2}} \, \|f\|^{|\mathbf{m}|}_{\frac{1}{H}-d}\, \mathbb{E}\Big(\prod_{i=1}^N \big(W(L_{b_i}(0)) - W(L_{a_i}(0)) \big)^{m_i} \Big), \tag{4.4}$$

where the expectation in the right-hand side of the above equation is given by formula (2.1).

Proof. For any $K > 0$ and $\ell = 1, \dots, |\mathbf{m}|/2$, we introduce the set

$$D^\ell_{|\mathbf{m}|,K} = \big\{ (s_1, \dots, s_{|\mathbf{m}|}) \in D_{\mathbf{m}} : \, s_1 < s_2 < \cdots < s_{|\mathbf{m}|},\, s_{2\ell} - s_{2\ell-1} > K \big\},$$

where $D_{\mathbf{m}}$ is defined in (4.3).

Taking into account the results proved in [3], the proof of the convergence (4.4) reduces to show that for all $\ell = 1, \dots, |\mathbf{m}|/2$

$$\lim_{K\to\infty} \limsup_{n\to\infty} n^{\frac{|\mathbf{m}|(Hd-1)}{2}}\, \mathbb{E}\Big(\int_{D^\ell_{|\mathbf{m}|,K}} \prod_{i=1}^{|\mathbf{m}|} f(B(s_i))\, ds \Big) = 0. \tag{4.5}$$

In order to prove (4.5), set $|\mathbf{m}| = 2m$. Then, it suffices to show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} n^{m(Hd-1)} \int_{\mathbb{R}^{2md}} \int_{D^\ell_{2m,K}} \Big(\prod_{i=1}^{2m} \widehat{f}(x_i) \Big) \exp\Big(-\frac{1}{2} \mathrm{Var}\Big(\sum_{i=1}^{2m} x_i \cdot B(s_i) \Big) \Big)\, ds\, dx = 0.$$

Using similar arguments as in the proof Proposition 3.1 we can write

$$\begin{aligned}
&\limsup_{n\to\infty} n^{m(Hd-1)} \int_{\mathbb{R}^{2md}} \int_{D^\ell_{2m,K}} \Big(\prod_{i=1}^{2m} \widehat{f}(x_i) \Big) \exp\Big(-\frac{1}{2} \mathrm{Var}\,(\sum_{i=1}^{2m} x_i \cdot B(s_i)) \Big)\, ds\, dx \\
=&\limsup_{n\to\infty} n^{m(Hd-1)} \int_{\mathbb{R}^{2md}} \int_{D^\ell_{2m,K}} \Big(\prod_{j=1}^{m} |\widehat{f}(z_{2j})|^2 \Big) \\
&\qquad\qquad \times \exp\Big(-\frac{1}{2} \mathrm{Var}\Big(\sum_{i=1}^{2m} z_i \cdot \big(B(s_i) - B(s_{i-1}) \big) \Big) \Big)\, ds\, dz.
\end{aligned}$$

The right hand side of the above equality is positive and less than or equal to

$$\limsup_{n\to\infty} n^{m(Hd-1)} \int_{\mathbb{R}^{2md}} \int_{D^\ell_{2m,K}} \Big(\prod_{j=1}^{m} |\widehat{f}(z_{2j})|^2 \Big) \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} |z_i|^2 (s_i - s_{i-1})^{2H} \Big)\, ds\, dz.$$

Integrating with respect all z_is and s_is ($i \neq 2\ell$), the above limit is less than or equal to

$$\begin{aligned}
&c_1 \int_{\mathbb{R}^d} \int_K^\infty |\widehat{f}(z_{2\ell})|^2 e^{-\frac{\kappa_H}{2} |z_{2\ell}|^2 u^{2H}}\, du\, dz_{2\ell} \\
\le\,& c_2 \int_{\mathbb{R}^d} \int_K^\infty |z_{2\ell}|^2 e^{-\frac{\kappa_H}{2} |z_{2\ell}|^2 u^{2H}}\, du\, dz_{2\ell} \\
=\,& c_3\, K^{1-Hd-2H}.
\end{aligned}$$

This completes the proof since $1 - Hd - 2H < 0$. ■

Proof of Theorem 1.1. This follows from Lemma 2.1, Propositions 4.1, 4.2 and 4.3 by the method of moments.

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